XXV. *On the Figures obtained by strewing Sand on Vibrating Surfaces, commonly called Acoustic Figures.* *By* CHARLES WHEATSTONE, *Esq.* *Communicated by* MICHAEL FARADAY, *Esq., D.C.L. F.R.S., &c. &c. &c.*

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§ 1.

HALF a century has nearly elapsed since the attention of philosophers was first called to the curious phenomena exhibited when sand is strewed on vibrating surfaces. Long before this time, GALILEO had noticed that small pieces of bristle laid on the sounding-board of a musical instrument, were violently agitated on some parts of the surface, whilst on other parts they did not appear to move; and our own countryman Dr. HOOKE, whose sagacity in anticipating many of the discoveries of later times has been so frequently remarked, had proposed to observe the vibrations of a bell by strewing flour upon it. But to CHLADNI is due the sole merit of having discovered the symmetrical figures exhibited on plates of regular forms when caused to sound. His first investigations on this subject, Entdeckungen über die Theorie des Klanges, were published in 1787; this work was followed by his Akustik in 1802, and his Neue Beyträge zur Akustik, 1817. A French translation, by himself, of his second work was published at Paris in 1809.

All the figures obtained by CHLADNI on square surfaces are delineated in pages 611, 613, 615; they are copied from the Neue Beyträge, which work contains his most mature experiments; but not having been translated either into French or English, it is but little known in this country. The following are the general results deduced by CHLADNI from his observations respecting these figures: his works may be referred to for the details omitted, and for those concerning the vibrations of plates in general.

In all the modes of vibration of a square or rectangular plate, the figures, even if they consist of diagonal or tortuous lines, may all be referred to a certain number of nodal lines in the two directions parallel to the sides.

To establish a convenient notation for these figures, he represents the lines in the two directions by numbers separated by a vertical line. Thus, for example, 3|0 signifies the mode of vibration, in which there are three lines in one direction and none in the other; 5|2 denotes that in which there are five lines parallel to one side, and two to the other, &c.

The nodal lines, which may always be considered as having been originally straight, may curve themselves more or less; and in general the flexions of these lines, whether they adjoin each other or are separated by a straight line, mutually approach to or recede from each other. In some modes of vibration the nodal lines are never straight.

In some instances, the same mode of vibration may manifest itself in two essentially different ways, according as the flexions of the lines, or the greater number of them, are inward or outward; in the first case, the sound is usually graver than in the second. This difference is remarked in those figures where there is an entire number of flexions, as in 2|0, 3|1, 4|0, 5|3, 6|2, &c.; but never in those figures where there are $1\frac{1}{2}$, $2\frac{1}{2}$, &c., as in 3|0, 4|1, 5|0, 5|2, &c. To distinguish the first from the second figure, CHLADNI places a horizontal line above the numbers in the first case, and below them in the second case, thus, $\overline{4|2}$, $\underline{4|2}$.

When two or more figures having the same notation, occur in the Table, the others are to be considered as distortions of the first, occasioned by altering the fixed points, and the place at which the bow is applied.

If four plates of the same size, and upon which the same figure has been produced, be placed together so as to form a larger square, this compound figure may also be more or less accurately produced on a single larger plate. Several instances of this may be seen by reference to the Table of figures.

The following Table contains the relative sounds (expressed both by their musical names and the number of their vibrations,) of all the modes of vibration of a square plate, experimentally ascertained by CHLADNI. The horizontal series of numbers denotes the lines parallel to one of the sides, and the vertical series those parallel to the other.

TABLE.

1		G, 6								
2	D−, 9− / E..F, 10+	B, 15	A^1..B^1 27+, 28−							
3	G♯, 25	B′, 30 / C^2+, 32? 33?	F♯^{2}, 45	C^3 + 64, 65						
4	G♯^{2}−, 49− / G♯^{2}, 50	B^2, −, 55, 56−	C♯^{3}, 70 / D^3, 72	F♯^{3} 90, 91	B^3−, 110+, 112					
5	E^3+, 81	F^3, 84 / F♯^{3}, 90 91	G♯^{3}, 98, 99, 100−	B^3, 119 120 / C^4, 125 126 128−	D♯^{4}, 150, 153	F♯^{4}, 180				
6	B^3, 120 121 / C^4−, 125 126	C^4, 128	C♯^{4}, 135 140 / D^4, 144	E^4, 160 162	G^4, 189 192 / G♯^{4}, 196 198 200−	B♭^{4}, 224+ 231−	C^5, 256 264			
7	F^4, 169	F^4 F♯^{4} 175? 180?	F♯^{4}+ 180+? 189−?	G♯^{4}+, 209 210 / A^4, 216 220	B^4, 240 240	C♯^{5}, 275 280 / D^5, 286 288	E^5, 320 324 325	F♯^{5}, 360 364		
8	B♭^{4}, 224 225 / B♭^{4}+225? 231?		B^4, 240 242 / C^5−245 250	C^5+, 256+ 264−	D^5, 286 288 / D♯^{5}, 294 299	F^5, 336 338	G^5, 377+ 384 / G♯^{5}, 390 392	A^5, 432 435	B^5, 480	
9	D^5, 289		E^5− 315	F^5 330 336 / F^5+, 343 345	F♯^{5}, 360	G♯^{5}−390 392	B♭^{5}, 450	C^5−, 495 / C^5 510 512	C♯^{6}+561	D♯^{6}, 612
	0	1	2	3	4	5	6	7	8	9

The figures placed as exponents indicate the octaves in which the sounds occur; and the characters + and − denote that the sounds to the characters of which they are affixed are respectively sharper and flatter than the true intervals.

Having thus briefly stated the general results deduced by CHLADNI from his experimental researches, I shall proceed to class and analyse the phenomena; and I shall endeavour to show, that all the figures of vibrating surfaces are the resultants of very simple modes of vibration, oscillating isochronously, and superposed upon each other; the resultant figure varying with the component modes of vibration, the number of the superpositions, and the angles at which they are superposed. In this first part of the investigation I shall confine myself to the figures of square and other rectangular plates.

§ 2.

The most simple modes of vibration of a rectangular surface are those which exhibit quiescent lines parallel to one of its edges. EULER has theoretically established, that a rod or band, having both its ends unfixed, can vibrate with 2, 3, 4, 5, 6, &c. quiescent lines parallel to the ends, and that the corresponding numbers of vibration are very nearly as the squares of the arithmetical progression 3, 5, 9, 11, &c. These conclusions are fully confirmed by experiment. He has proved, moreover, that when the same mode of vibration of different plates is compared, the number of vibrations is inversely as the square of the length of the plate, but that increase of breadth occasions no difference in the sound; and that the distance from a free end to a quiescent line is rather less than half the distance between two quiescent lines.

Fig. 1. *a.* page 617, shows the number and situations of the quiescent lines in the first four modes of vibration of this series. Fig. 1. *b.* and *c.* are profiles of the preceding, and represent the curvature of each parallel fibre perpendicular to the quiescent lines at the two opposite limits of their vibration. The quantity of motion at each point is indicated by the corresponding ordinate of the curve, and its direction by its situation above or below the horizontal line. It will be convenient to distinguish these states of motion, in which every corresponding point is moving in direct opposition; and I shall therefore call the first, *b.* positive states of vibration, and the second, *c.* negative states of vibration. When there is an even number of quiescent points, the positive state of vibration may be considered as that in which the motion at the *central* part is above the plane of equilibrium, and the negative, that in which it is below it. If we suppose two similar surfaces with the same number of quiescent lines to

be superposed, and both to vibrate in concurrence, i. e. both either positively or negatively, they will mutually assist each other's effects; but if they vibrate in opposing directions, they will destroy each other's motions, and the entire surface will be at rest.

§ 3.

When the rectangular surface is equilateral, it is obvious that it may vibrate in two different rectangular directions, so as to give the same sound, and present the same arrangement of quiescent lines. Now this plate may be excited at various points where the motion of each mode of vibration is at its maximum, in the same direction, and of equal intensity: such being the case, there is no reason why one mode of vibration should be produced in preference to the other; and on calculating the effect of such coexistence, it will be found that the resultants of these combined modes of vibration, similar in everything but in their direction with regard to the sides of the plate, give rise to new quiescent lines which accurately correspond with figures described by CHLADNI; while the number of vibrations does not materially differ from that of the component modes of vibration.

The principal results of the superposition of two similar modes of vibration are these: 1st, The points where the quiescent lines of each figure intersect each other, remain quiescent points in the resultant figure; 2ndly, The quiescent lines of one figure are obliterated when superposed by the vibrating parts of the other; 3rdly, New quiescent points, which may be called points of compensation, are formed wherever the vibrations in opposite directions neutralize each other; and, lastly, At all other points the motion is as the sum of the concurring, or the difference of the opposing vibrations.

A primary figure, having an even number of quiescent lines, may be superposed two ways, and may consequently give rise to two distinct resultant figures: one, when the central vibrating parts concur; and the other, when they are in opposition; but if the number of the quiescent lines in the primary figure be uneven, there can be only one resultant figure.

The quiescent lines which thus result may be very easily ascertained. I will take as an example the first mode of vibration, having two parallel quiescent lines; this being superposed in two rectangular directions, and so that the states of vibration are opposing (page 617, fig. 2.), it is obvious that no lines of compensation can exist in the four rectangular segments *a a a a*, as every point included within them is actuated by concurrent motions; but in all the other rectangles they must necessarily be formed, as every point within them is affected by the two opposing motions, and if the two modes be of equal intensity, the compensations must occur at every point equally distant from the two rectangular quiescent lines, each appertaining to a different mode of vibration. The resultant figure will thus be found to consist of two diagonal lines perpendicular to each other, and passing through the centre of the plate.

But if the two superpositions vibrate in concurrence, the rectangles *b b b b* will be free from compensating points; but these will occur in the other rectangles, and form the figure represented (fig. 3.), which also consists of diagonal lines.

In the same manner the resultant of any two similar modes of vibration with nodal lines, parallel to the sides, may be proved to consist of lines parallel to the diagonals.

§ 4.

It is not a necessary condition for the vibrations of a square plate that the primary nodal lines shall be parallel to a side; they may also be parallel to a diagonal, or to any line intermediate between a transverse and a diagonal line. In these cases the superpositions take place according to the following rule: That the axes of the superposed modes of vibration must make equal angles with a transversal line passing through the centre; for otherwise the modes of vibration would not be similar. By the axis of a primary mode of vibration, I mean a straight line passing through the centre of the plate and parallel to the quiescent lines. Considerations of the kind already employed will show that in all these instances the resultant figures consist of lines parallel to the edges of the plate, and that they are always the same in number as the nodal lines of a component mode of vibration, but differently distributed in the two directions, according as the angle of superposition varies.

The various primary modes of vibration, transverse, intermediate, and diagonal, and the angles which the quiescent lines of two similar figures make with each other when they are superposed, are represented in the first column of the general Table, page 619—633; in the second column of this Table are placed the figures resulting from their opposing superpositions, and in the third column those which arise from their concurring superpositions.

We obtain by experiment a limited number only of figures which can be considered the resultants of primary modes of vibration consisting of any given number of oblique lines; but it would seem, that as the various degrees of obliquity are infinite, so there should be an infinite number of resultant figures passing into each other by insensible gradations: by calculation this should be so, but there are causes of limitation which I shall proceed to explain.

It appears that no resultant figure is maintainable unless the greatest excursions of the external vibrating parts occur at the edges of the plate. In the concurring superpositions of eight oblique lines, this condition can only be fulfilled when the angles they make with each other are either 90°, or 143° 8′; in the first case the resultant figure consists of four lines in each transverse direction, in the second of six lines in one direction and two in the other. In the opposing superpositions of the same number of lines, the condition is fulfilled when the angles at which the lines are inclined are 118° 4′ and 163° 44′; the resultant figure of the former consists of five lines in one direction and three in the other, and that of the latter of seven in one direction and one in the other.

§ 5.

I have, in the preceding sections, described the various modes of binary superpositions which may take place on a square surface. But there are numerous cases in which four superpositions may coexist, and these I shall now proceed to take into consideration.

When the axis of a primary figure corresponds either with a diagonal or with a transverse line, passing through the centre of the plate, it is obvious that there can be only one other line of equal length, which can be considered as the axis of a similar and isochronous mode of vibration; in these cases it is evident, therefore, that there can only be two superpositions. But in every intermediate direction of an axis, there are three other lines of equal length which constitute axes of similar modes of vibration; and four superpositions can therefore take place whenever the axis of a component mode of vibration is neither a diagonal nor a transverse line.

It would be a tedious and laborious process to ascertain a resultant figure by combining its four component modes of vibration; but the same purpose will be effectually answered by combining them first in pairs, as explained in the preceding section, and then combining two of these first resultants rectangularly together.

The following process affords great facility for ascertaining the second resultant figure, which arises from the superposition of two first resultants. *a.* and *b.* (fig. 4. page 617,) are the two component first resultants, the similar lines of one being placed rectangularly to those of the other; the vibrating parts are indicated by the letters P and N, according as the vibrations are positive or negative. At C the two figures are superposed, A being represented by the continuous and B by the dotted lines. The surface is now subdivided into a number of unequal rectangles, and by comparing the two component figures together, it is easy to see which of these rectangles are influenced by conspiring, and which by opposing motions; if the motions are found to conspire, the letters P or N must be placed in these rectangles according as the coexisting motions are positive or negative; if the motions are in opposition, a mark may be made to indicate that a quiescent line passes through this rectangle. Wherever a continuous line intersects a dotted line, a mark is to be made, to indicate that a quiescent point is formed; and as in every other part, the quiescent lines of one figure pass over vibrating parts of the other, the boundary lines of all the rectangles must be marked with the letters indicating the motions of the vibrating parts they superpose. The figure C being thus marked, the resultant figure is easily described by joining the fixed points by lines drawn through the rectangles shown to be actuated by opposing motions; carefully avoiding to encroach upon the rectangles of conspiring motion marked P or N.

That the diagonal line is perfectly straight may be proved in the following manner. It must first be premised that the rectangles included within the quiescent lines of each of the first resultant figures, have precisely the same quantity of motion in the same relative points with respect to the surrounding sides, and that the vibrating parts at the edges and corners of the plate must be considered respectively as exact halves and quarters of a complete vibrating part. This being understood, if two similar first resultants be laid alongside each other in the directions in which they are to be superposed, and if a diagonal line be similarly drawn through each of them, it will be obvious that every corresponding point of each line must possess the same intensity of motion. If the successive segments (equal in each figure,) through which the lines simultaneously pass, be in opposite states of vibration, they will neutralize each other's effects, and a diagonal quiescent line will be formed; and if they be concurring, all the parts between the coincident or fixed points will be in motion. In the diagram (fig. 4. *c.*) one diagonal is in the first state and the other in the second.

If the number of lines in the component first resultant figures be uneven, they admit of only one mode of superposition. But first resultants having an even number of quiescent lines, admit of being superposed in two ways, according as they are vibrating in concurrence or in opposition.

It frequently occurs that entire quiescent lines superpose each other; thus, when in the component figure there is an uneven number of lines in each direction, the two rectangular central lines of each superposed figure must coalesce, and consequently they continue in both the resultant figures; examples of this are seen in the resultants of 3|1, 5|1, 5|3, 7|1, 7|3, 7|5, &c. Again, when the number of quiescent lines in one direction of the component figure is three times greater than that in the other, the number of coinciding fixed lines in each direction of the resultant figure is equal to the smallest number in the first resultant, as in 6|2, 9|3, 12|4, 15|5, &c. See the general Table, page 619—633.

The following general results are obtained from constructing the second resultant figures according to the rules above given.

In superposing first resultants consisting of an even number of lines: 1st, When the number of lines in each direction of the first resultants is even, and the modes of vibration are concurring, no line passes through the centre of the second resultant figure. When the modes of vibration are opposing, two rectangular diagonal lines of compensation occur. 2ndly, When the number of lines in each direction is uneven, and the modes of vibration are concurring, there are always two perpendicular transversal lines passing through the centre; and when the modes are opposing, there are, in addition to these fixed lines, the two diagonal lines of compensation.

When the first resultants consist of an uneven number of lines, in which case there is no distinction of concurring and opposing vibrations, one diagonal line only invariably occurs.

In no case is it necessary to calculate an entire figure. When the number of nodal lines in the primary mode of vibration is even, only one quarter of the figure is required to be calculated, as it is obvious that every second resultant of this kind consists of four symmetrical, and as it were reflected, portions. But when the primary number of quiescent lines is uneven, it is necessary to calculate one half; the other half is symmetrical and inverted.

Some of the first resultants are never obtained by experiment. When the number of quiescent lines in the primary mode of vibration is uneven, either the first or the second resultant may be obtained at pleasure; thus in 3|2 if the impulses be made at a corner, where the motion of both superpositions is at its maximum, the second resultant must arise; but if they be made at the middle of a side, the first resultant only will appear, because the point of excitation is a quiescent point of the other. But in all cases where there is an *even* number of lines, it is impossible to obtain the first resultant, because each maximum of vibration equally belongs to both superpositions.

§ 6.

I have given in pages 619—633, a Table which shows every perfect resultant figure of a square surface when the number of quiescent lines in the primary modes of vibration does not exceed twelve. I have carefully calculated each figure by the rules laid down in the preceding sections, and have shown in the Table the successive processes of superposition. The first vertical row exhibits the two primary modes of vibration superposed at the required angles; one figure being represented by the continuous lines, and the other by the dotted lines. The second row contains the first resultants which arise from the opposing superpositions of the preceding; and the third row, those which result from their concurring superpositions. The fourth and fifth rows exhibit the perfect second resultants which are formed, the former by the opposing and the latter by the concurring superpositions of the first resultant which the plate has been already found competent to produce.

On comparing the calculated figures with those obtained experimentally by Chladni, the greater number are found exactly to agree; there are, however, some differences which it will be necessary to explain. In the first place, there is an obvious cause of error in delineating figures from experiment, from this circumstance,—that the sand accumulates in the spaces where two convex curves are near and opposite to each other, the motion being there very small, so that it is difficult to ascertain whether the curves join, or not. Secondly, Inequalities in the plate will sometimes occasion lines which ought to intersect each other, so as to appear separated curves. On comparing together the figures CHLADNI has marked as $\underline{6|4}$, $\overline{6|4}$, 7|2, 7|4, 8|3 *c.* &c. with those of the calculated Table, they will be found to differ only in these respects.

Another cause of difference is this: When the lines of one component figure very nearly coincide with those of the other, but without actually doing so, the resultant figure may be such as would arise from their actual superposition, instead of that which accurate calculation would give. In CHLADNI's Table there are two instances of this alteration, 7|2 *a.* and 8|3 *a.*

A few of the figures delineated by CHLADNI are irregular resultants formed by the superpositions of dissimilar modes of vibration. These irregular resultants can be formed only when the dissimilar component modes of vibration give the same sound, and have a maximum point of vibration in common, at which they can be simultaneously excited. The figure marked by CHLADNI 6|1, I find to be an irregular resultant formed by the combination of 6|1 with 3|5, which both give the same sound C^4. The irregular figure marked in his Table 5|1 *a.* is a compound of 5|1 and 2|5.

The calculated figures 6|1, 7|1, 8|1, 9|1, 10|1, 10|2, and 11|1, are not to be found in CHLADNI's Table. The near approach of the inclined lines of their primary component figures to parallelism is the cause of the great difficulty in obtaining these figures by experiment.

The figures marked by CHLADNI 10|3, 9|4, 8|5, 7|6, 10|4, 9|5, 8|6, and 9|7, exceed the limits within which I have calculated the Table, and are therefore not to be found in it.

§ 7. *Imperfect resultant Figures.*

I have hitherto considered those resultant figures only which arise from the superpositions of similar modes of vibration, each exactly equal in intensity; these I shall in future call *perfect* resultant figures. But when the vibrations of the superposed modes are unequal in intensity, then a figure intermediate between the perfect resultant and one of its components is formed; these intermediate figures I shall call *imperfect* resultants. They are experimentally obtained by varying in a slight degree the places at which the plate is held or touched, from those necessary to determine the corresponding perfect resultant figure; the place at which the bow is applied remaining in both cases the same.

Fig. 6. *a. b. c. d. e.* in CHLADNI's Table, page 611, represents the successive

transformations of figure which take place when each component mode of vibration presents three transversal lines: *a.* and *e.* are the two components; *c.* the perfect resultant; *b.* an imperfect resultant, in which the excursions of *a.* are the greatest: and *d.* an imperfect resultant, in which *e.* has the greatest energy.

Fig. 11. *a.* *b.* page 611, exhibits the transitions of the opposing superposition of two primary figures, each presenting four transversal lines; and fig. 12. *a.* and *b.* the changes of the concurring superposition of the same.

These are the principal types of the transformation of primary figures into first resultants. In each of these series of transitions there are certain points which are invariable during every change: these are, the quiescent points formed by the nodal lines of one figure intersecting those of the other, and the centres of vibration where the maxima of positive or negative vibration agree in each component mode of vibration. The points of compensation are changeable.

Figs. 29. *a.* and 30. *a.* page 613, represent imperfect second resultants, formed by two superpositions of the first resultant figure 6|2. Fig. 29. *a.* arises from concurring, and fig. 30. *a.* from opposing superpositions. The straight lines in these imperfect figures arise from the coincidence of entire quiescent lines in each component figure, and, consequently, they remain unaltered whatever may be the relative intensities of the superposed modes of vibration. But the curved lines, which are formed of compensating points, change with the varying intensities.

§ 8. *Figures of irregular Plates.*

If the sides of the square be nearly, but not exactly, equal, the superpositions of two similar modes of vibration with transversal lines still take place; but instead of exhibiting perfect resultants, figures resembling transitional figures appear. Thus in the binary superposition (page 611, fig. 1.) of the figure with two transversal lines, if the sides be unequal, the crossed lines separate at their point of intersection and are converted into two curves, the summits of which recede from each other as the difference in the lengths of the sides becomes greater.

Also, if the diagonals of the square be unequal, the resultant figure (page 611, fig. 2.) arising from two superposed modes of vibration with diagonal axes, will be modified in a similar manner.

Corresponding modifications are occasioned, through accidental differences of elasticity, &c. in the directions of the axes of the superposed modes of vibration, even when the dimensions of the plate are apparently equal.

If a plate of glass be covered on one of its sides with leaf-gold, or if a plate of ground glass be substituted for an ordinary glass plate with smooth surfaces, the figures may be obtained distinctly delineated by lines consisting of a single row of grains of sand. Experiments made in this manner, upon square plates carefully prepared, induced Professor STREHLKE to conclude, after many minute measurements of these lines, that all acoustic figures are formed of hyperbolic curves, and that the quiescent lines never intersect each other. But however correct these experiments may have been, the conclusions drawn from them are unwarranted; were it possible to obtain plates of a perfectly homogeneous substance, and of accurately equal dimensions, there can be no doubt that the lines, however finely defined, would actually intersect each other.

§ 9.

I have already, § 1., given CHLADNI's Table of the comparative sounds, and numbers of vibrations of the figures of square surfaces obtained from experiment. In the following Table these results are arranged so as to correspond with the views taken in this paper. The numbers in the first vertical row indicate the number of parallel quiescent lines in the primary figure. In the horizontal rows the angles at which the lines of the primary modes of vibration intersect each other are shown; and the figures between brackets give the notation of the first resultant produced by their superposition; below these its number of vibrations, and the character representing its musical sound, are given; when there are two of these lower lines separated by a horizontal dash, the one above indicates the sound of the opposing superposition, and that below it the concurring superposition.

Thus it appears, that every figure of a square surface which experiment can give, may be reduced to a primary figure with parallel lines, giving the same sound: if, therefore, the analytical investigation be confined to these, many of the difficulties will disappear. EULER has investigated the subject when lines parallel to a side only are concerned; it remains to extend the inquiry to modes of vibration the nodal lines of which are perpendicular to any line passing through the centre of the surface. An analytical expression for all the sounds of a square plate may probably be obtained, which shall be a function of the number of quiescent lines, and the length of the axis of the mode of vibration.

TABLE.

2	90° (1|1) G, 6	180° (2|0) D—, 9 ――― E..F, 10+					
3	126° 52′ (2|1) B, 15	180° (3|0) $G\sharp^{1}$, 25					
4	90° (2|2) A^{1}..B^{1}, 27+, 28—	143° 8′, (3|1) B^{1}, 30 ――― C^{2}+, 32.?.33.?	180° (4|0) $G\sharp^{2}$—, 49— ――― $G\sharp^{2}$, 50				
5	112° 38′, (3|2) $F\sharp^{2}$, 45	151° 56′, (4|1) B^{2}—, 55..56—	180°, (5|0) E^{3}+, 81				
6	90°, (3|3) C^{3}+, 64.65	126° 52′, (4|2) $C\sharp^{3}$, 70 ――― D^{3}, 72	157° 22′ (5|1) F^{3}, 84 ――― $F\sharp^{3}$, 90.91	180° (6|0) B^{3}, 120..121 ――― C^{4}—, 125..126			
7	106° 16′ (4|3) $F\sharp^{3}$, 90..91	136° 22′ (5|2) $G\sharp^{3}$, 98.99.100—,	161° 4′ (6|1) C^{4}, 128	180° (7|0) F^{4}, 169			
8	90° (4|4) $B\flat^{3}$—, 110+.112	118° 4′, (5|3) B^{3}.119.120 ――― C^{4}, 125.126.128—	143° 8′ (6|2) $C\sharp^{4}$ 135.140 ――― D^{4}.144	163° 44′ (7|1) F^{4}..$F\sharp^{4}$ 175.? 180.?	180° (8|0) $B\flat^{4}$ 224.225 ――― B^{4}+225+231.?		
9	102° 40′ (5|4) $D\sharp^{4}$ 150.153	126° 52′ (6|3) E^{4} 160.162	148° 6′ (7|2) $F\sharp^{4}$+.180+.?189—	165° 44′ (8|1)	180° (9|0) D^{5} 289		
10	90° (5|5) $F\sharp^{4}$, 180	112° 38′ (6|4) G^{4} 189..192 ――― $G\sharp^{4}$ 196.198.200—	133° 26′ (7|3) $G\sharp^{4}$+ 209.210 ――― A^{4}, 216.220	151° 56′ (8|2) B^{4} 240.242 ――― C^{5}— 245.250	167° 18′ (9|1) unascertained.	180° (10|0)	
11	100° 24′ (6|5) $B\flat^{4}$, 224+, 231—	120° 30′ (7|4) B^{4} 240.242	138° 52′ (8|3) C^{5}+, 256+.264—	154° 56′ (9|2) E^{5}— 315	168° 34′ (10|1)	180° (11|0)	
12	90° (6|6) C^{5} 256+.264—	108° 56′ (7|5) $C\sharp^{5}$, 275.280 ――― D^{5}, 286.288	126° 52′ (8|4) D^{5}, 286.288 ――― $D\sharp^{5}$ 294.299	143° 8′ (9|3) F^{5}, 330.336 ――― F^{5}+ 343, 345	157° 22′ (10|2)	169° 36′ (11|1)	180° (12|0)

§ 10.

Immediately after the publication of CHLADNI's experiments on square plates, JAMES BERNOUILLI attempted to demonstrate them analytically; but his investigation was entirely unsuccessful; his conclusions were founded on erroneous data, and the results he obtained were at variance with experiment. His assumptions were these: that the primary figures consisted of 2, 3, 4, 5, 6, 7, &c. lines parallel to the side only, these being the modes of vibration of a lamina as investigated by EULER; that any two similar or dissimilar modes of vibration might superpose each other rectangularly, the nodal lines of the two components appearing together in the resultant mode of vibration; and that the sound of the resultant differed from that of either of the two components, being much higher. The figures given by experiment he considered accidental distortions of these compound figures. This theory gave no account of those figures in which a single line in one direction coexists with any number in the other, as there is no primary figure consisting of one nodal line only; and BERNOUILLI acknowledged his theory to be imperfect in this respect.

The failure of BERNOUILLI led CHLADNI inconsiderately to state, that "the supposition of regarding such a rigid membraniform body as a network formed by curved lines in one direction applied upon curved lines in another direction, is not conformable to nature, and will never give, either results agreeing with experiment, or an appearance of explanation of some of the most simple vibrations." That this assertion is erroneous, the considerations in the present paper have, I conceive, fully proved. The error of BERNOUILLI did not consist in assuming that the observed acoustic figures were formed by superposing simple modes of vibration on each other, for this has been shown to be true; but in his assumptions of the manner in which these superpositions were made, and the effects which resulted from such hypothetical superpositions.

The various mathematicians who have more recently undertaken to investigate the laws of vibrating surfaces, as POISSON, CAUCHY, Mademoiselle GERMAIN, &c., do not appear to have taken into consideration anything resembling the theory of superposition.

Dr. YOUNG seems to have had a correct notion of the origin of the acoustic figures; for in his Lectures, when slightly noticing CHLADNI's experiments, he remarks, "The vibrations of plates differ from those of rods in the same manner as the vibrations of membranes differ from those of chords, the vibrations which cause the plate to bend in different directions being combined with each other, and sometimes occasioning singular modifications."

The brothers WEBER, in their excellent work the Wellenlehre, published in 1825, have advanced a step nearer the truth than any of their predecessors. They have shown, that in a square vessel containing water or mercury, two series of stationary waves, one parallel to each side, may be made to intersect each other; and that the compound wave formed by their interferences assumes the form shown in § 3. to be the resultant of two superpositions of parallel transverse lines. Their observations are confined to those modes of undulation, analogous to the first resultants of primary modes of vibration with lines parallel to a side. Though I had advanced considerably in the present inquiry before I saw this work, yet I should be wanting in justice to these philosophers did I not here state that theirs is the merit of having shown, in the most simple case, the way in which the superpositions of modes of vibration or undulation actually do take place.

§ 11. *Plates of Wood.*

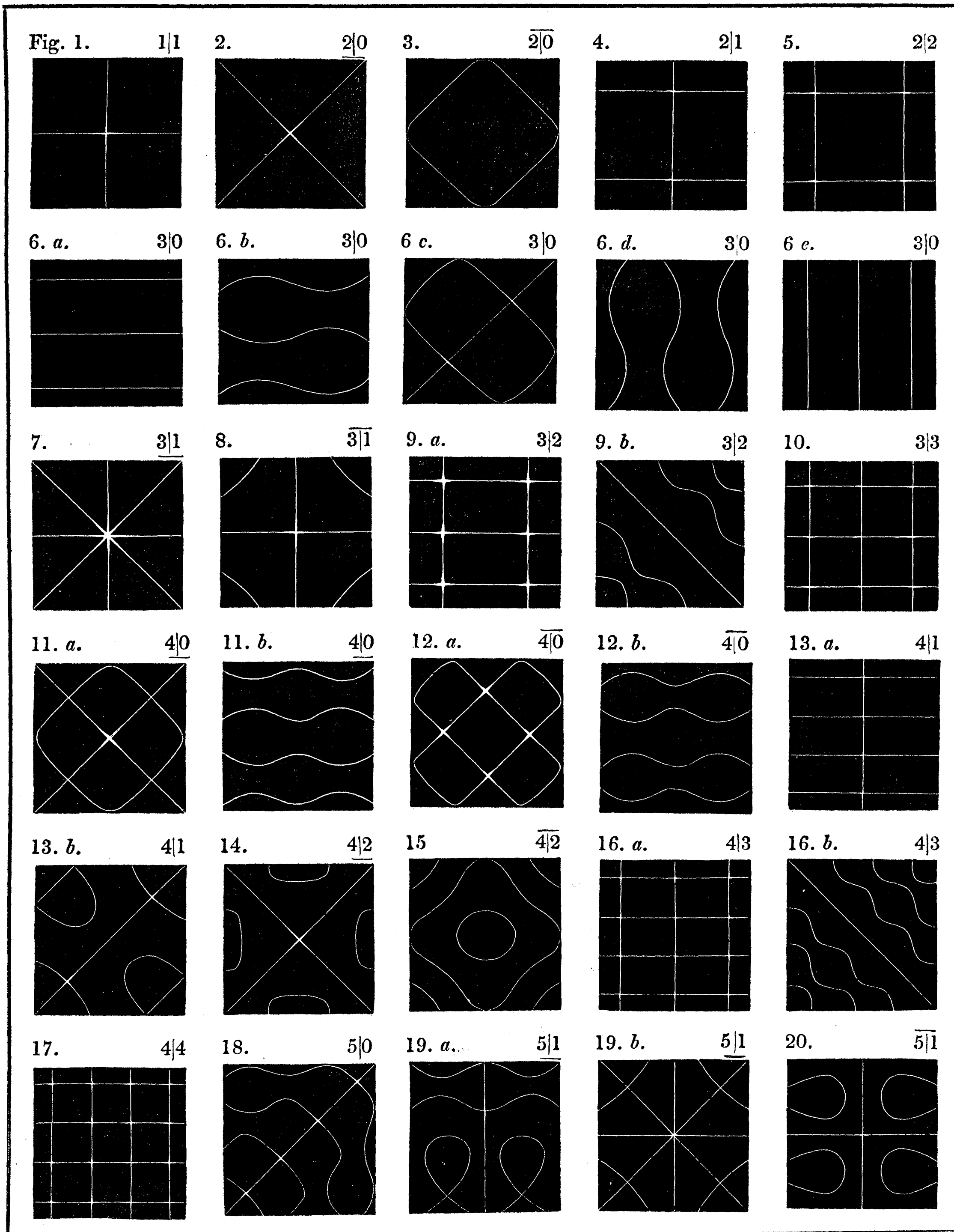
From the rules already laid down, it is obvious that the series of figures presented by a square plate of any homogeneous material ought not to be obtained on a square plate of wood, in which substance the elasticity is not the same in all directions. If a square plate of wood be prepared with its fibres parallel to one of the sides of the square, the axes of greatest and least elasticity will be disposed rectangularly, and parallel to the adjacent sides; in this case the same primary mode of vibration in the two directions will not give the same sound, although the dimensions of the vibrating parts are the same in both; consequently they cannot coexist, and the resultant figures with diagonal lines will be wanting on such a plate. But if the axes of the two component modes of vibration be equally inclined to either of the axes of elasticity, these modes of vibration will be necessarily similar and isochronous, and therefore capable of superposition; on a square plate of wood, consequently, all those first resultants which consist of any number of lines parallel to the sides intersecting each other rectangularly, may be obtained; and the same figure will be accompanied by different sounds, according as the axes of the modes of vibration are inclined to the axes of least or greatest elasticity. It is easy to foresee that none of the second resultants, which consist of four isochronous superpositions, can be obtained on such a plate.

But if the wooden plate, instead of being square, be a rectangle, the sides of which are inversely as the squares of their resistance to flexion, the two modes of vibration parallel to the sides, though differing in length, will be isochronous; and their coexistence will give rise to a resultant figure with lines parallel to the diagonal. Thus on a rectangular plate of straight-fibred deal wood, in which the proportion of the sides were as 28 to 59, I obtained the two crossing diagonal lines, corresponding to the second figure of a homogeneous square plate of glass or metal.

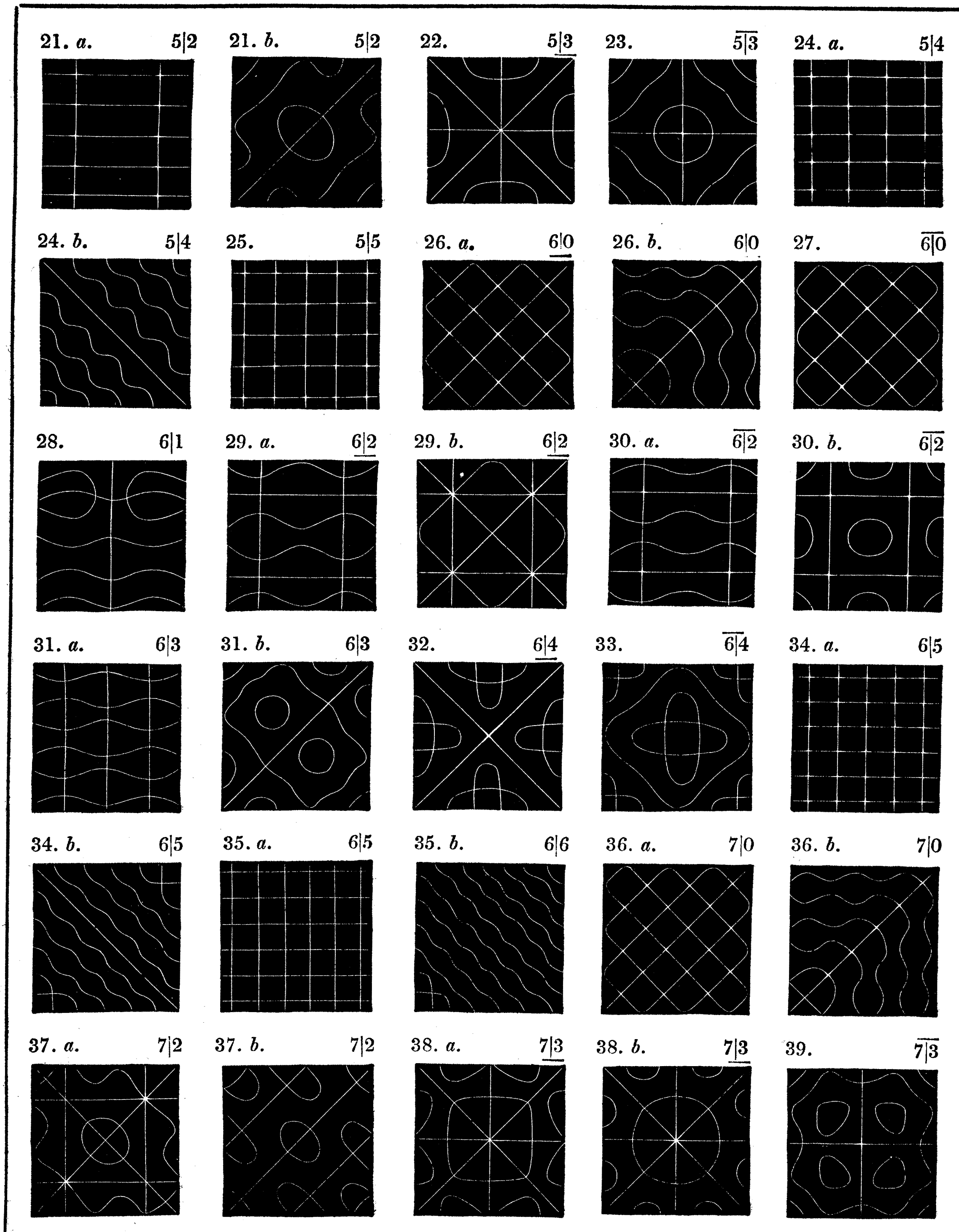
Savart has made a series of numerous and accurate experiments on the changes which take place in the sound, and also in the form and position of the figure of the first mode of vibration on circular plates of wood of similar dimensions, cut in different directions with respect to the three principal axes of elasticity. All the results he has obtained are in perfect accordance with the rules stated in this paper, and might have been predicted by them. He has extended his investigations to circular slices of crystals, cut in various directions with respect to their axes, and has obtained in this way much valuable information. These researches of Savart point out a new direction to our inquiries respecting the structure of bodies; and the utility of his experiments will be greatly enhanced by the knowledge we now possess of the causes on which these phenomena depend. I shall shortly return to this subject.

Chladni's Table

Of the figures of square surfaces, obtained by experiment.



(Continuation of Chladni's Table.)



(Continuation of Chladni's Table.)

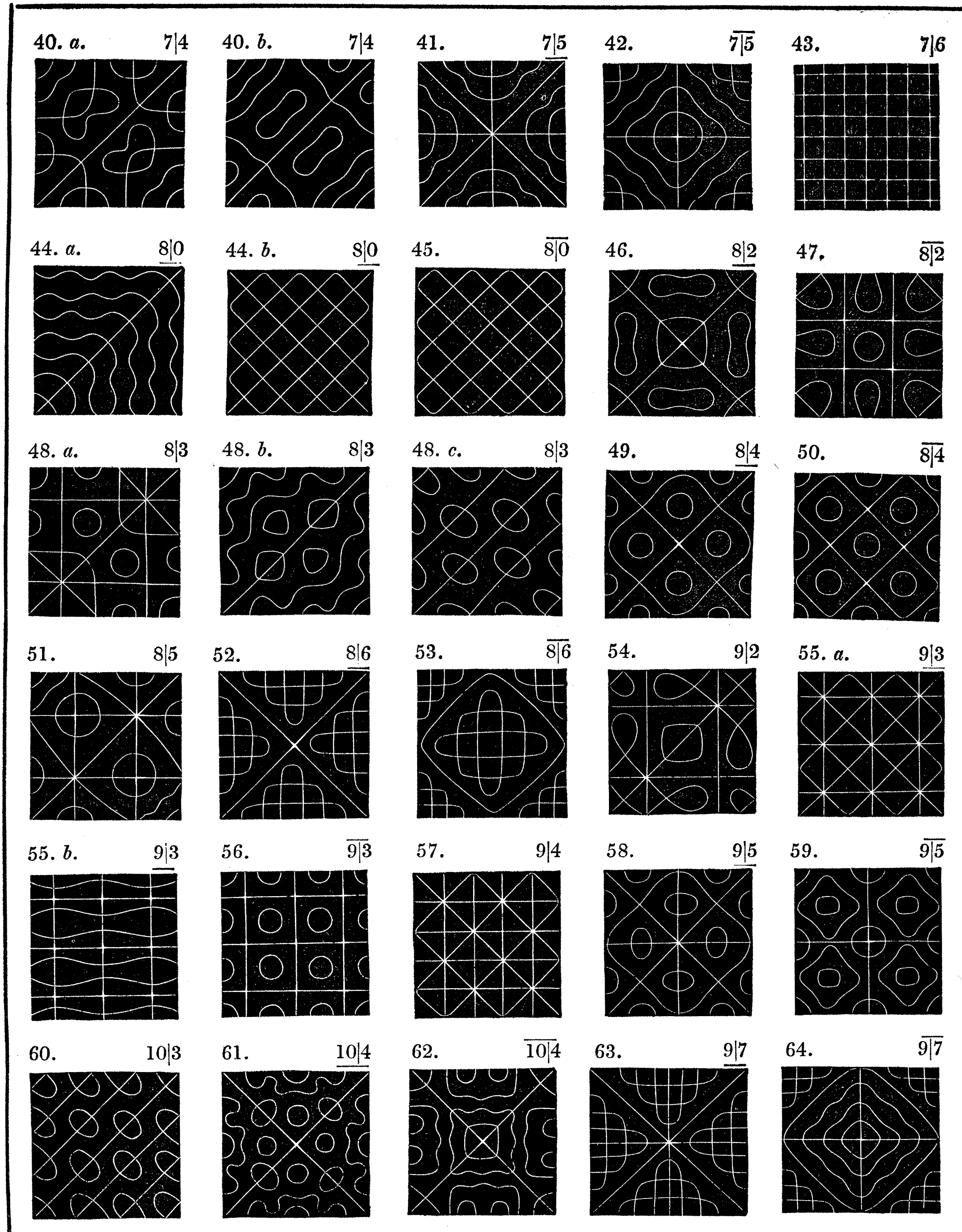


Fig. 1.

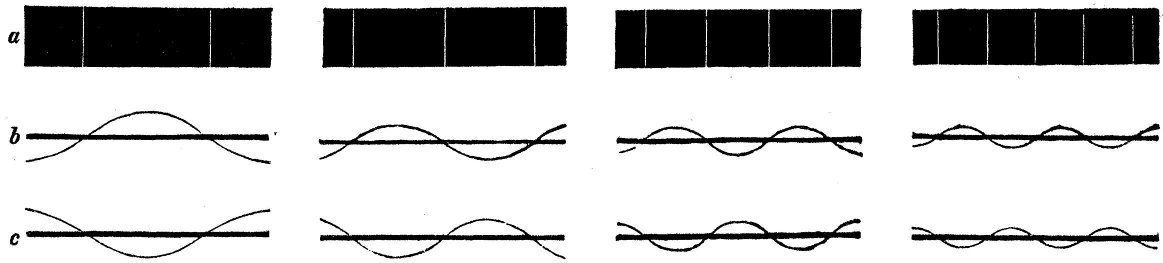


Fig. 2.

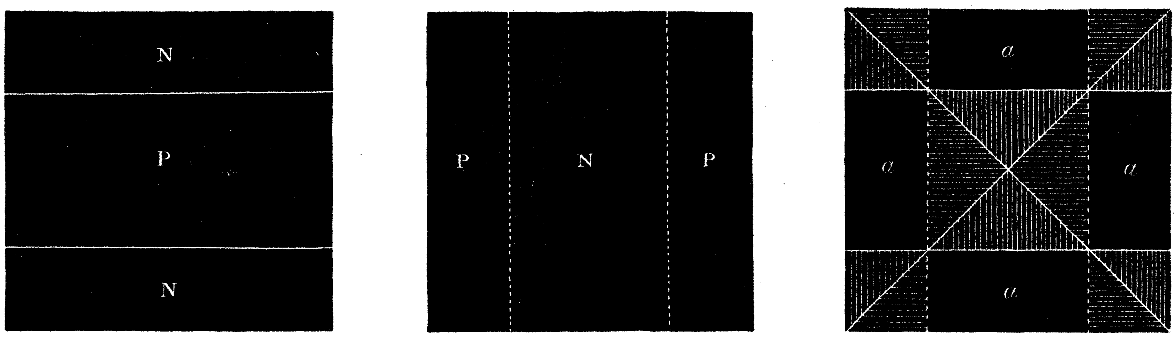


Fig. 3.

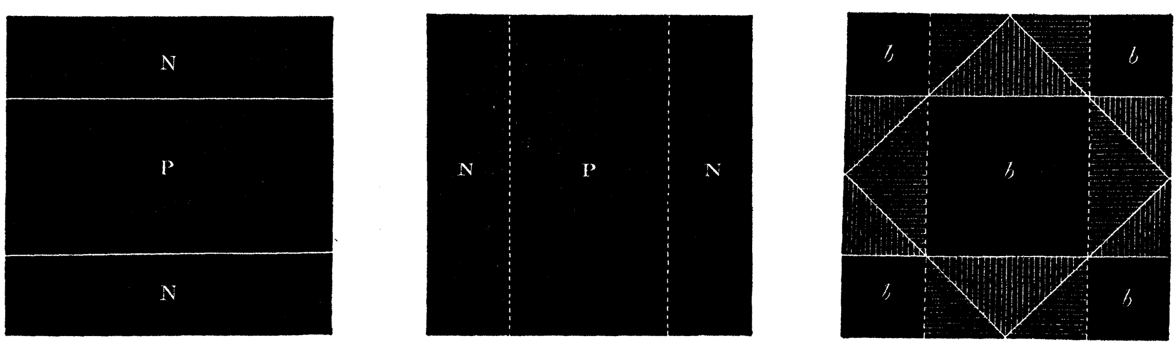
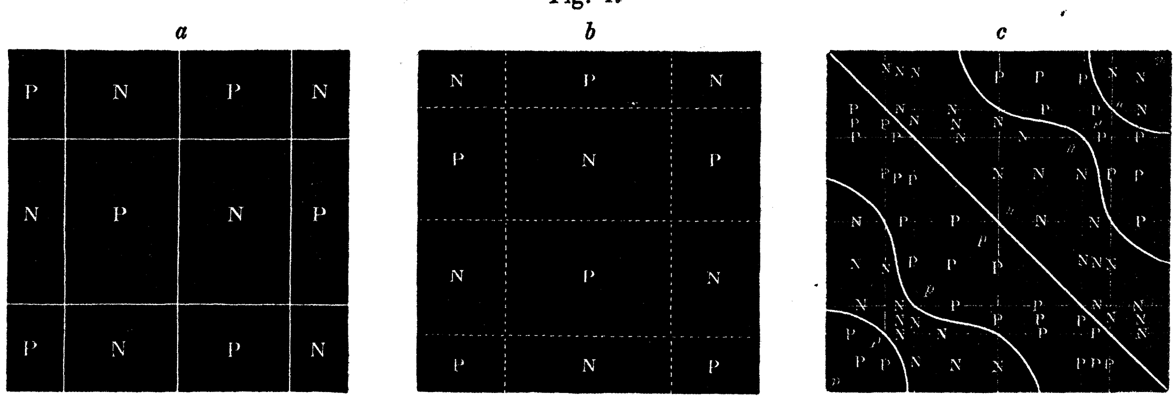


Fig. 4.



Tabular View

Of all the figures of a square plate, determined by calculation to result from two or four superpositions, when the quiescent lines of the primary figures do not exceed twelve in number.

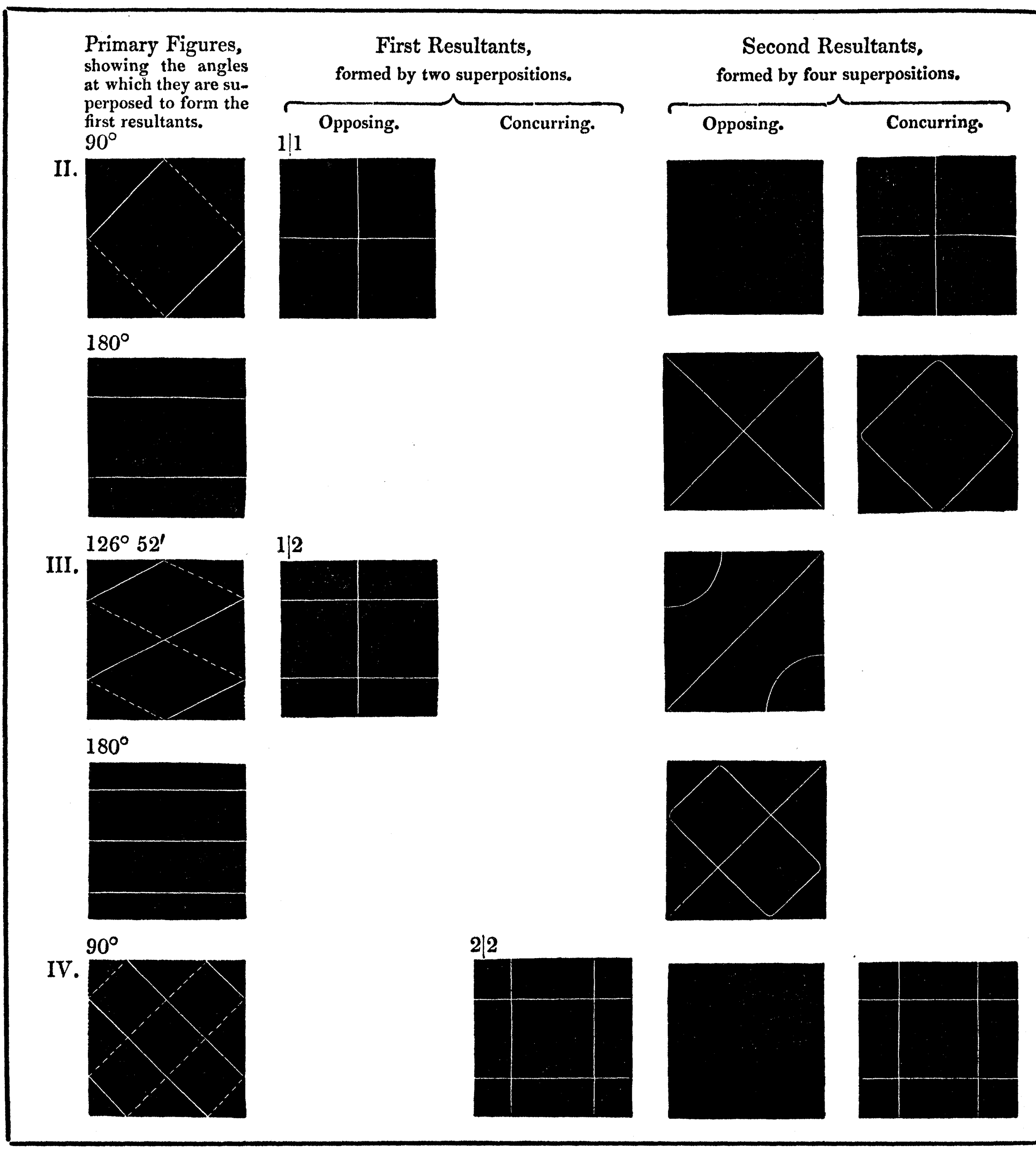


TABLE of Calculated Figures.—(Continued.)

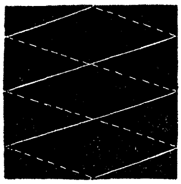
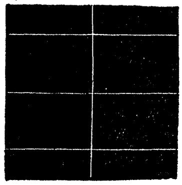
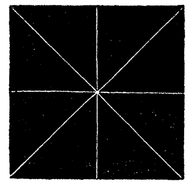
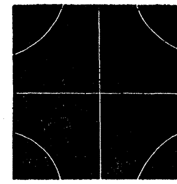
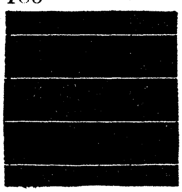
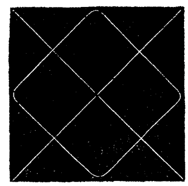
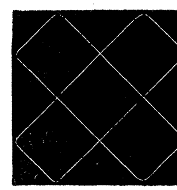
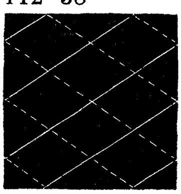
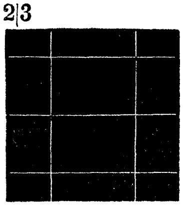
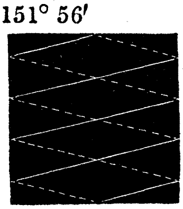
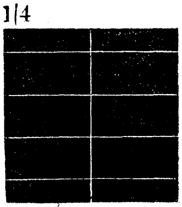
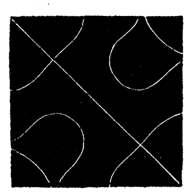
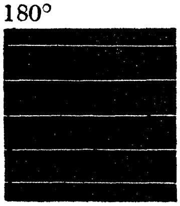
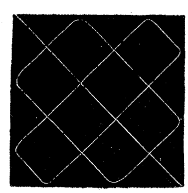
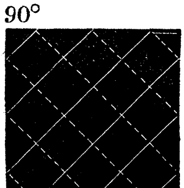
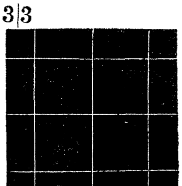
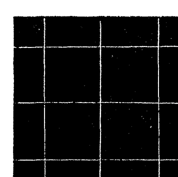
	Primary Figures, showing the angles at which they are superposed to form the first resultants.	First Resultants, formed by two superpositions.		Second Resultants, formed by four superpositions.	
		Opposing.	Concurring.	Opposing.	Concurring.
	143° 8′ 	1|3 			
	180° 				
V.	112° 38′ 	2|3 			
	151° 56′ 	1|4 			
	180° 				
VI.	90° 	3|3 			

TABLE of Calculated Figures.—(Continued.)

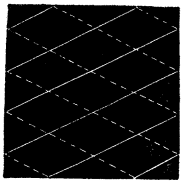
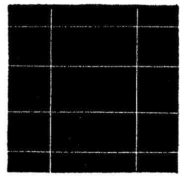
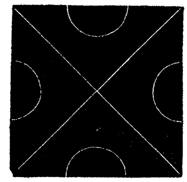
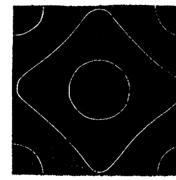
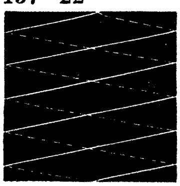
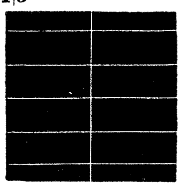
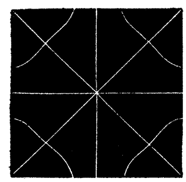
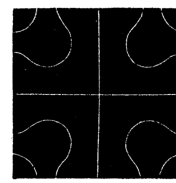
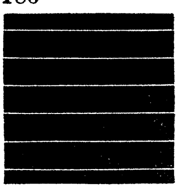
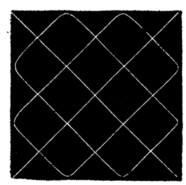
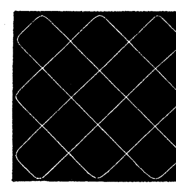
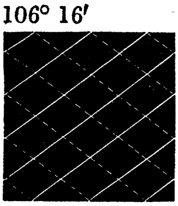
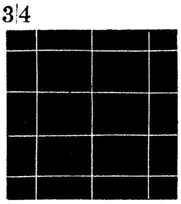
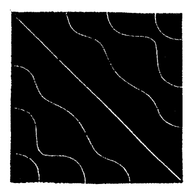
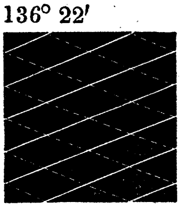
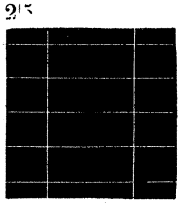
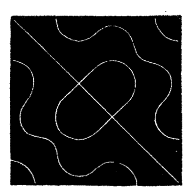
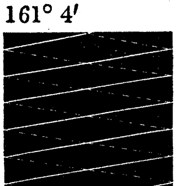
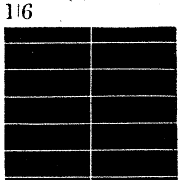
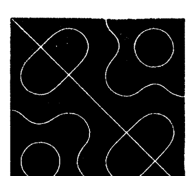
	Primary Figures, showing the angles at which they are superposed to form the first resultants.	First Resultants, formed by two superpositions. Opposing.	First Resultants, formed by two superpositions. Concurring.	Second Resultants, formed by four superpositions. Opposing.	Second Resultants, formed by four superpositions. Concurring.
	126° 52′ 		2|4 		
	157° 22′ 	1|5 			
	180° 				
VII.	106° 16′ 	3|4 			
	136° 22′ 	2|5 			
	161° 4′ 	1|6 			

Table of Calculated Figures.—(Continued.)

Primary Figures, showing the angles at which they are superposed to form the first resultants. | First Resultants, formed by two superpositions. (Opposing. / Concurring.) | Second Resultants, formed by four superpositions. (Opposing. / Concurring.)

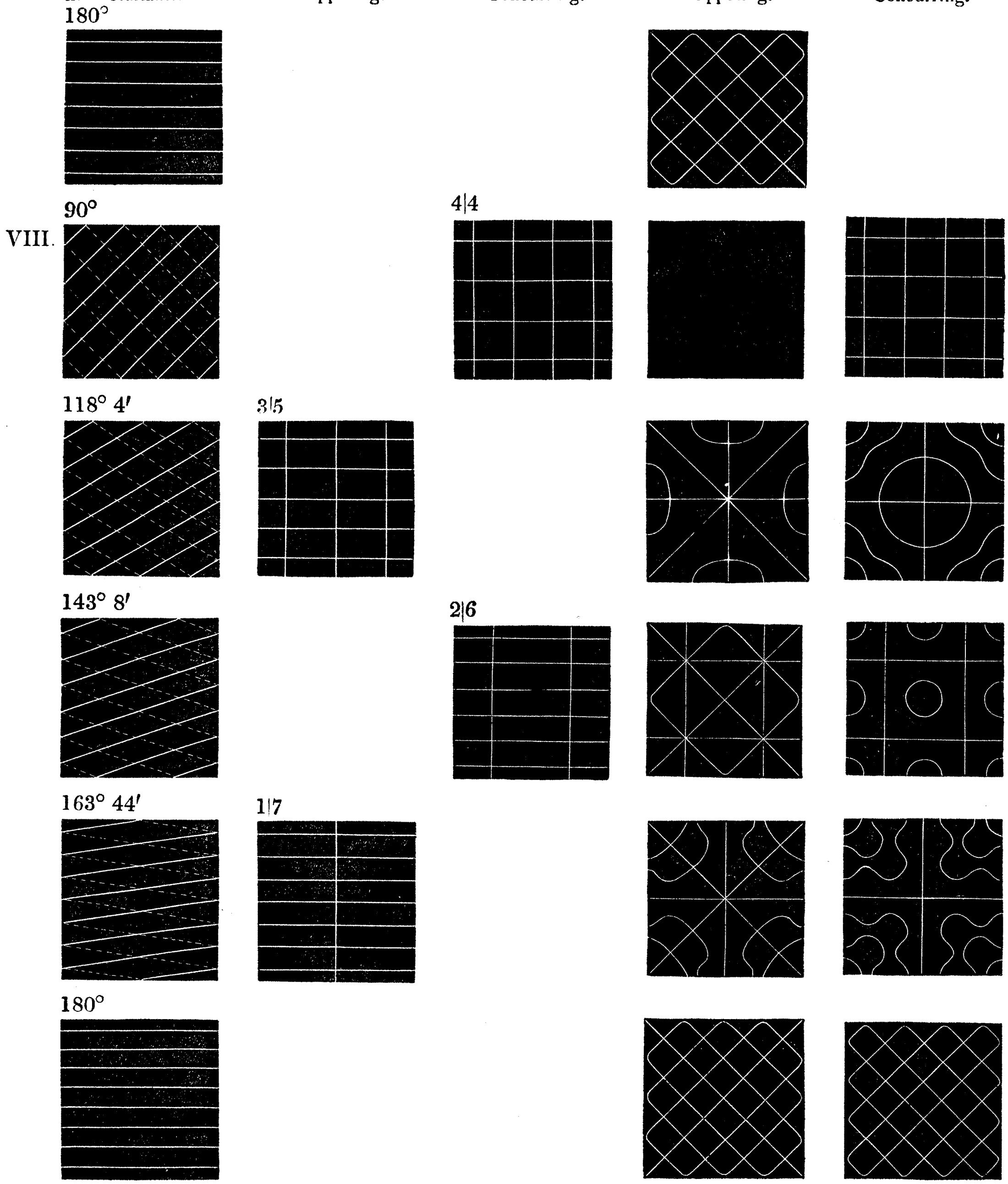


Table of Calculated Figures.—(Continued.)

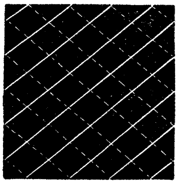
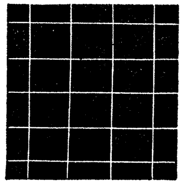
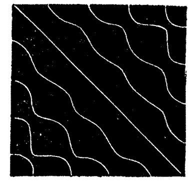
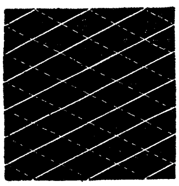
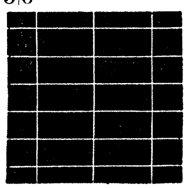
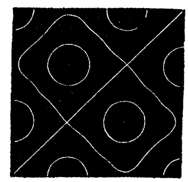
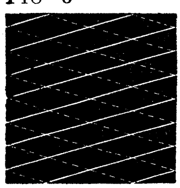
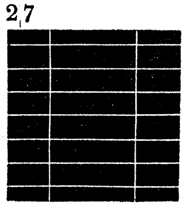
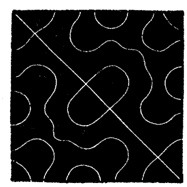
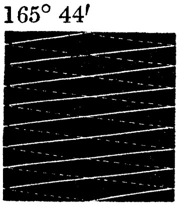
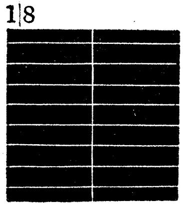
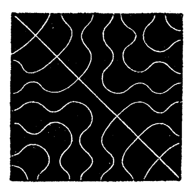
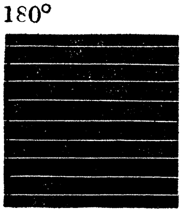
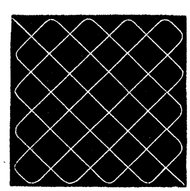
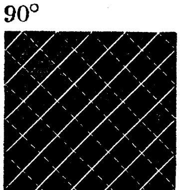
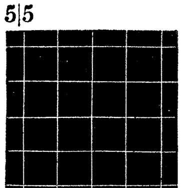
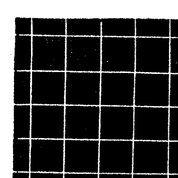
	Primary Figures, showing the angles at which they are superposed to form the first resultants.	First Resultants, formed by two superpositions. Opposing.	First Resultants, formed by two superpositions. Concurring.	Second Resultants, formed by four superpositions. Opposing.	Second Resultants, formed by four superpositions. Concurring.
IX.	102° 40′ 	4|5 			
	126° 52′ 	3|6 			
	148° 6′ 	2|7 			
	165° 44′ 	1|8 			
	180° 				
X.	90° 	5|5 			

TABLE of Calculated Figures.—(Continued.)

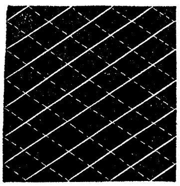
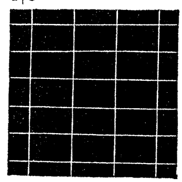
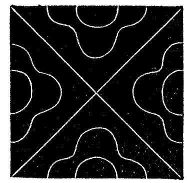
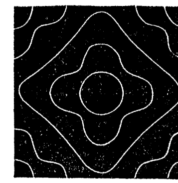
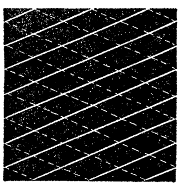
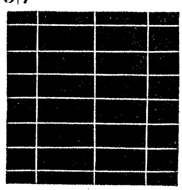
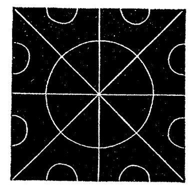
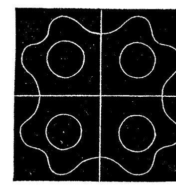
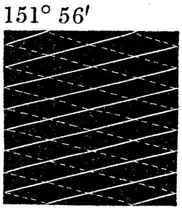
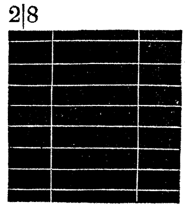
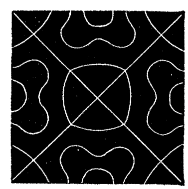
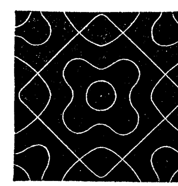
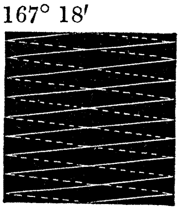
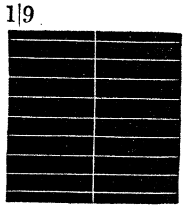
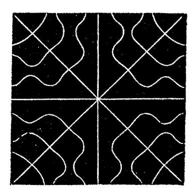
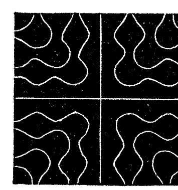
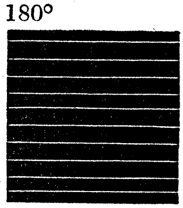
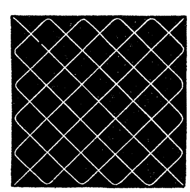
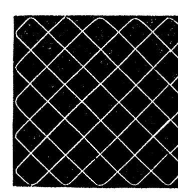
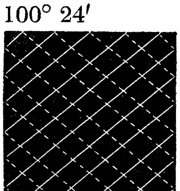
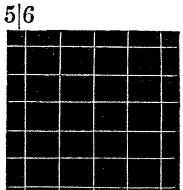
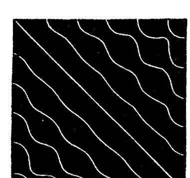
Primary Figures, showing the angles at which they are superposed to form the first resultants.	First Resultants, formed by two superpositions. Opposing.	First Resultants, formed by two superpositions. Concurring.	Second Resultants, formed by four superpositions. Opposing.	Second Resultants, formed by four superpositions. Concurring.
112° 38′ 		4|6 		
133° 26′ 	3|7 			
151° 56′ 		2|8 		
167° 18′ 	1|9 			
180° 				
XI. 100° 24′ 	5|6 			

Table of Calculated Figures.—(Continued.)

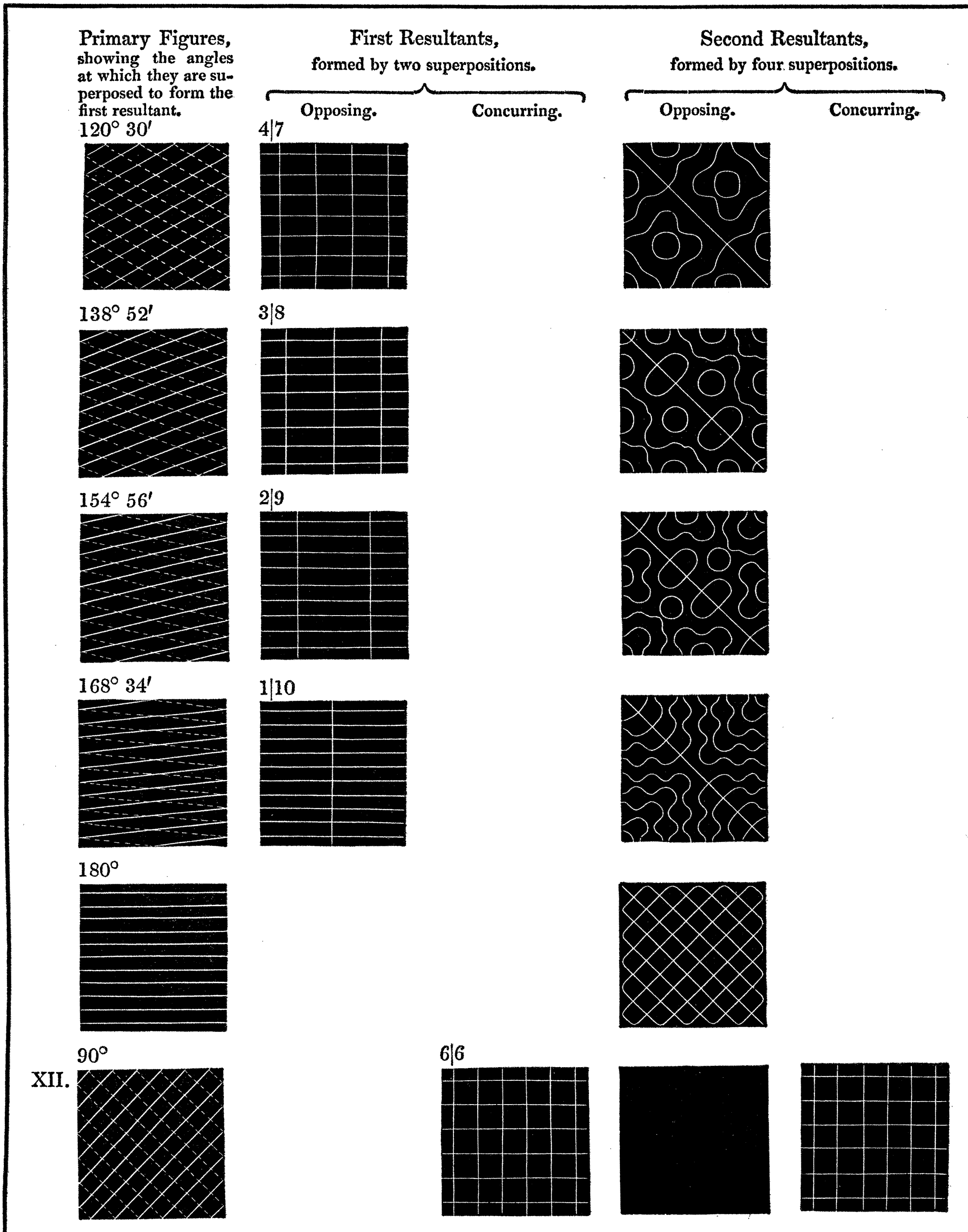
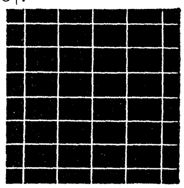
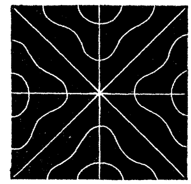
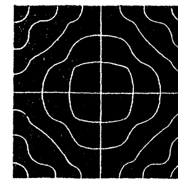
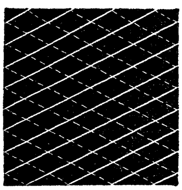
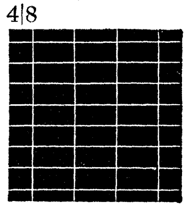
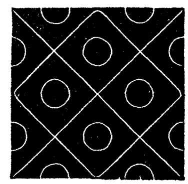
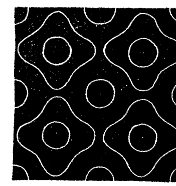
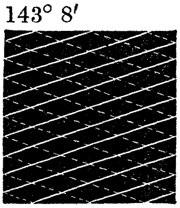
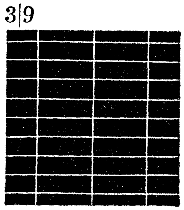
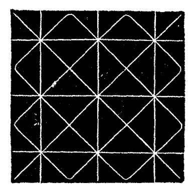
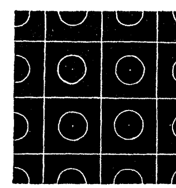
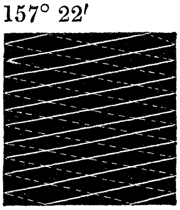
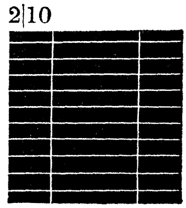
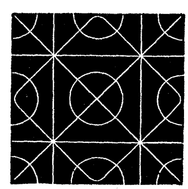
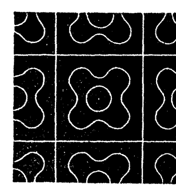
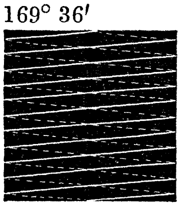
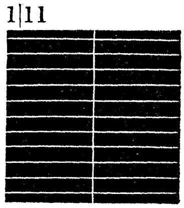
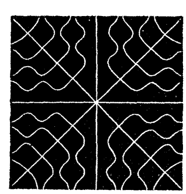
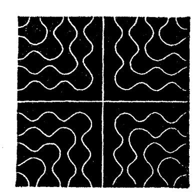


Table of Calculated Figures.—(Continued.)

Primary Figures, showing the angles at which they are superposed to form the first resultants.	First Resultants, formed by two superpositions.		Second Resultants, formed by four superpositions.	
	Opposing.	Concurring.	Opposing.	Concurring.
108° 56′ 	5|7 			
126° 52′ 		4|8 		
143° 8′ 	3|9 			
157° 22′ 		2|10 		
169° 36′ 	1|11 			
180°